PARALLEL AND PERPENDICULAR LINES IN THREE DIMENSIONS

1 Given L_1 has equation $x = 1 + 2t$, $y = 2 - t$, $z = 3 + t$ and L_2 has equation $x = 2 + s$, $y = -1 + 2s$, $z = 1 - 3s$, then L_1 and L_2:

A are parallel **B** intersect **C** are perpendicular **D** are skew

* A vector parallel to L_1 is $2\vec{i} - \vec{j} + \vec{k} = \vec{V_1}$

A vector parallel to L_2 is $\vec{i} + 2\vec{j} - 3\vec{k} = \vec{V_2}$

$\frac{2}{1} \neq \frac{-1}{2} \neq \frac{1}{-3}$ ∴ these two vectors are NOT parallel

∴ L_1 and L_2 are not parallel

* For L_1 and L_2 to intersect, we must have

$$\begin{cases} 1+2t = 2+s \\ 2-t = -1+2s \\ 3+t = 1-3s \end{cases} \Leftrightarrow \begin{cases} 2t = s+1 & ① \\ -t = 2s-3 & ② \\ t = -3s-2 & ③ \end{cases}$$

By elimination for ② and ③, we obtain $0 = -s - 5$ so $s = -5$

So $t = -3 \times (-5) - 2 = 13$

But from ① $2 \times 13 \neq -5 + 1$ so the lines do NOT intersect.

* For the two lines to be perpendicular, the scalar product of their vectors must be 0. Let's check:

$\vec{V_1} \cdot \vec{V_2} = 2 \times 1 + (-1) \times 2 + 1 \times (-3) = 2 - 2 - 3 \neq 0$

So the two lines are NOT perpendicular.

* The lines are NOT parallel, NOT perpendicular, and don't intersect.

∴ are skew

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4 Given L_1 has equation $x = 2 + 2t$, $y = 2 + t$, $z = 3 - t$ and L_2 has equation $x = 2 + s$, $y = -1 + 2s$, $z = -6 + 4s$, then L_1 and L_2:

A are parallel **B** intersect **C** are perpendicular **D** are skew

* For L_1 $\vec{V_1} = 2\vec{i} + \vec{j} - \vec{k}$

For L_2 $\vec{V_2} = \vec{i} + 2\vec{j} + 4\vec{k}$

$\frac{2}{1} = 2$ whereas $\frac{1}{2} = \frac{1}{2}$ and $-\frac{1}{4} = -\frac{1}{4}$

So L_1 and L_2 are NOT parallel.

* For the lines to intersect, we must have

$$\begin{cases} 2+2t = 2+s \\ 2+t = -1+2s \\ 3-t = -6+4s \end{cases} \Leftrightarrow \begin{cases} 2t = s & (1) \\ t = 2s-3 & (2) \\ -t = 4s-9 & (3) \end{cases}$$

For (2) and (3), by elimination, we obtain $0 = 6s - 12$ $\boxed{s = 2}$

So $t = 2 \times 2 - 3 = 1$

Substituting in (1), we obtain indeed $2 \times 1 = 2$

So the lines intersect at $(4,3,2)$

* $\vec{V_1} \cdot \vec{V_2} = 2 \times 1 + 1 \times 2 - 1 \times 4 = 2 + 2 - 4 = 0$

So the lines are perpendicular.

* lines are perpendicular and intersect, ∴ not skew.

5 Line L_1 passes through the points (1, 2, −1) and (4, −1, 2) while line L_2 passes through the points (2, 6, −2) and (a, −1, 5), where $a \in R$.

Find the value(s) of a, if:

(a) L_1 is parallel to L_2 **(b)** L_1 is perpendicular to L_2.

a) For L_1 and L_2 to be parallel, we must have $\vec{V_1} = \lambda \vec{V_2}$ with $\lambda \in \mathbb{R}$ $\vec{V_1} = 3\vec{i} - 3\vec{j} + 3\vec{k} = 3(\vec{i} - \vec{j} + \vec{k})$

$\vec{V_2} = (a-2)\vec{i} - 7\vec{j} + 7\vec{k} = 7\left[\left(\frac{a-2}{7}\right)\vec{i} - \vec{j} + \vec{k}\right]$

So we must have $\frac{a-2}{7} = 1$ or $a - 2 = 7$ $\boxed{a = 9}$

b) For the lines to be perpendicular, we must have

$\vec{V_1} \cdot \vec{V_2} = 0$

$\vec{V_1} \cdot \vec{V_2} = 3(a-2) + (-3)\times(-7) + 3\times 7$

$— = 3a - 6 + 21 + 21$

$— = 3a + 36$

So we must have $3a + 36 = 0$

$3a = -36$

$\boxed{a = -12}$

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7 Find the coordinates of the points where the line $\underset{\sim}{r} = (1-\lambda)\underset{\sim}{i} + (4+2\lambda)\underset{\sim}{j} + (3-\lambda)\underset{\sim}{k}$ cuts the coordinate planes.

* x-y plane : $z=0$ so $3-\lambda=0 \Rightarrow \boxed{\lambda=3}$
  Point is $(-2, 10, 0)$
* y-z plane : $x=0$ so $1-\lambda=0$ or $\boxed{\lambda=1}$
  Point is $(0, 6, 2)$
* x-z plane : $y=0$ so $4+2\lambda=0$ $2\lambda=-4$ $\boxed{\lambda=-2}$
  Point is $(3, 0, 5)$

α **9 (a)** Show that the line (L_1) through the points (2, 1, −1) and (3, −2, 0) is parallel to the line (L_2) through the points (−2, −3, 2) and (2, −15, 6).

(b) Show that the point (1, 4, −2) lies on the first line and the point (6, −27, 10) lies on the second line.

a) $\vec{V_1} = \vec{i} - 3\vec{j} + \vec{k}$ whereas $\vec{V_2} = 4\vec{i} - 12\vec{j} + 4\vec{k}$

So $\vec{V_2} = 4(\vec{i} - 3\vec{j} + \vec{k}) = 4\vec{V_1}$

$\therefore$ L_1 and L_2 are parallel

b) let $A(1, 4, -2)$

The vector equation of the 1st line is

$$\vec{V} = 2\vec{i} + \vec{j} - \vec{k} + \lambda(\vec{i} - 3\vec{j} + \vec{k})$$
$$\vec{V} = (2+\lambda)\vec{i} + (1-3\lambda)\vec{j} + (-1+\lambda)\vec{k}$$

For $\lambda = -1$, we have $\begin{cases} 2+\lambda = 1 \\ 1-3\lambda = 4 \\ -1+\lambda = -2 \end{cases}$ so A belongs to L_1

Let $B(6, -27, 10)$. The vector equation of the 2nd line is:

$$\vec{u} = -2\vec{i} - 3\vec{j} + 2\vec{k} + \lambda(4\vec{i} - 12\vec{j} + 4\vec{k})$$
$$\vec{u} = (-2+4\lambda)\vec{i} + (-3-12\lambda)\vec{j} + (2+4\lambda)\vec{k}$$

For $\lambda = 2$ we have $-2+4\lambda = 6$ ✓ $-3-12\times2 = -27$ ✓

and $2+4\times2 = 10$ ✓

So B belongs to the 2nd line

10 Show that the line through the points (2, 1, −1) and (3, −2, 3) is perpendicular to the line through the points (1, 3, 2) and (−1, 5, 4).

L_1 L_2

For L_1 $\quad \vec{V_1} = \vec{i} - 3\vec{j} + 4\vec{k}$

for L_2 $\quad \vec{V_2} = -2\vec{i} + 2\vec{j} + 2\vec{k}$

$\vec{V_1} \cdot \vec{V_2} = 1 \times (-2) + (-3) \times 2 + 4 \times 2$

$\phantom{\vec{V_1} \cdot \vec{V_2}} = -2 - 6 + 8$

$\phantom{\vec{V_1} \cdot \vec{V_2}} = 0$

The dot product is zero, $\therefore$ $\vec{V_1}$ and $\vec{V_2}$ are $\perp$

So the lines are perpendicular.

PARALLEL AND PERPENDICULAR LINES IN THREE DIMENSIONS

11 **(a)** Find the equation of the line L_1 through the point (2, 1, −2) parallel to the vector $\underset{\sim}{v} = \underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$.

(b) Find the equation of the line L_2 through the points (1, −2, 1) and (0, 2, −2).

(c) Determine whether **(i)** $L_1 \parallel L_2$, **(ii)** $L_1 \perp L_2$, **(iii)** L_1 and L_2 intersect.

a) $\vec{V_1} = 2\vec{i} + \vec{j} - 2\vec{k} + \lambda\overbrace{(\vec{i} - 2\vec{j} + 3\vec{k})}^{=\vec{u_1}}$

$\vec{V_1} = (2+\lambda)\vec{i} + (1-2\lambda)\vec{j} + (-2+3\lambda)\vec{k}$

which is the equation of L_1

b) $\vec{V_2} = \vec{i} - 2\vec{j} + \vec{k} + \lambda\overbrace{(-\vec{i} + 4\vec{j} - 3\vec{k})}^{=\vec{u_2}}$

$\vec{V_2} = (1-\lambda)\vec{i} + (-2+4\lambda)\vec{j} + (1-3\lambda)\vec{k}$

That's the equation of line L_2.

c) i) $\vec{u_1} \neq x\vec{u_2}$ $\therefore$ L_1 not parallel to L_2

ii) $\vec{u_1} \cdot \vec{u_2} = 1 \times (-1) + (-2) \times 4 + 3 \times (-3)$

$= -1 - 8 - 9$

$\neq 0$ so not perpendicular

iii) They intersect if $\exists\, \lambda, t \in \mathbb{R}$ such that $\begin{cases} 2+\lambda = 1-t \\ 1-2\lambda = -2+4t \\ -2+3\lambda = 1-3t \end{cases} \Leftrightarrow \begin{cases} \lambda = -t-1 & (1) \\ -2\lambda = 4t-3 & (2) \\ 3\lambda = -3t-3 & (3) \end{cases}$ or $\lambda = -t-1$

So equations (1) and (3) are identical.

By elimination for solving (1) and (2) : $\begin{cases} 2\lambda = -2t-2 \\ -2\lambda = 4t-3 \end{cases}$

So $0 = 2t - 5$ $\quad \boxed{t = 5/2}$ $\quad \lambda = -\frac{5}{2} - 1 = -7/2$ $\quad \boxed{\lambda = -7/2}$

For L_1, that's $\left(-\frac{3}{2}, 3, -\frac{25}{2}\right)$

For L_2 that's $\left(-\frac{3}{2}, 8, -\frac{13}{2}\right)$. so no intersection as the 2 points are different.