COMPLEX NUMBERS - CHAPTER REVIEW

1 If $z = 1 + 2i$ and $w = -3 - 4i$, find the following in $x + iy$ form:

(a) $3z + w$ **(b)** z^2 **(c)** $w\overline{w}$ **(d)** $\frac{z}{w}$ **(e)** the square roots of w.

a) $3z + w = 3(1+2i) + (-3-4i) = 3 - 3 + 6i - 4i = 2i$

b) $z^2 = (1+2i)^2 = 1 + 4i - 4 = -3 + 4i$

c) $w\overline{w} = (-3-4i)(-3+4i) = 9 + 16 = 25$

d) $\frac{z}{w} = \frac{1+2i}{(-3-4i)} = \frac{(1+2i)(-3+4i)}{(-3-4i)(-3+4i)}$

$\frac{z}{w} = \frac{-3+4i-6i-8}{25} = -\frac{11}{25} - \frac{2i}{25}$

e) $\sqrt{w} = \sqrt{-3-4i}$

So we look for $(a+ib)$ such that $(a+ib)^2 = -3-4i$

$\begin{cases} a^2 - b^2 = -3 \\ 2ab = -4 \end{cases}$ $\begin{cases} a^2 - b^2 = -3 \\ ab = -2 \end{cases}$

Now $(a^2+b^2)^2 = (a^2-b^2)^2 + (2ab)^2$

$(a^2+b^2)^2 = 9 + 16 = 25$

so $a^2 + b^2 = 5$

So $2a^2 = 2$ $a^2 = 1$ $a = \pm 1$

$ab = -2$ so $b = \mp 2$

$\sqrt{w} = \pm(1-2i)$

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6 **(a)** Evaluate the following, giving answers in both mod–arg form and $x + iy$ form.

(i) $(\sqrt{3} - i)^3$ **(ii)** $\dfrac{(1-\sqrt{3}i)^2}{(1+i)^3}$

(b) Use your answer to part **(a)(ii)** to show that $\cos\dfrac{7\pi}{12} = \dfrac{\sqrt{2}-\sqrt{6}}{4}$.

a) i) $(\sqrt{3}-i)^3 = \left[2\left[\frac{\sqrt{3}}{2} - \frac{i}{2}\right]\right]^3 = 2^3 \times \left(e^{-i\pi/6}\right)^3 = 8\,e^{-i\pi/2} = -8i$

ii) $\dfrac{(1-\sqrt{3}i)^2}{(1+i)^3} = \dfrac{\left[2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right]^2}{\left[\sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)\right]^3} = \dfrac{2^2\left[e^{-i\pi/3}\right]^2}{(\sqrt{2})^3\left(e^{i\pi/4}\right)^3} = \dfrac{4}{2\sqrt{2}}\,\dfrac{e^{-i2\pi/3}}{e^{i3\pi/4}}$

$= \dfrac{2}{\sqrt{2}}\,e^{i\left[-\frac{2\pi}{3} - \frac{3\pi}{4}\right]} = \sqrt{2}\,e^{-i\frac{17\pi}{12}} = \sqrt{2}\,e^{i\frac{7\pi}{12}}$

But $\dfrac{(1-\sqrt{3}i)^2}{(1+i)^3} = \dfrac{1 - 2\sqrt{3}i - 3}{(1+i)(1-1+2i)} = \dfrac{-2\sqrt{3}i - 2}{2i(1+i)} = \dfrac{-2-2\sqrt{3}i}{-2+2i}$

So $\dfrac{(1-\sqrt{3}i)^2}{(1+i)^3} = \dfrac{2+2\sqrt{3}i}{2-2i} = \dfrac{1+\sqrt{3}i}{1-i} = \dfrac{(1+\sqrt{3}i)(1+i)}{1+1} = \dfrac{1-\sqrt{3}+i(1+\sqrt{3})}{2}$

So $\dfrac{(1-\sqrt{3}i)^2}{(1+i)^3} = \left[\dfrac{1-\sqrt{3}}{2}\right] + i\left[\dfrac{1+\sqrt{3}}{2}\right]$

b) $\therefore \sqrt{2}\,e^{i\frac{7\pi}{12}} = \left[\dfrac{1-\sqrt{3}}{2}\right] + i\left[\dfrac{1+\sqrt{3}}{2}\right]$

So, equalling real parts, we obtain:

$\sqrt{2}\cos\left(\dfrac{7\pi}{12}\right) = \dfrac{1-\sqrt{3}}{2}$ or $\cos\left(\dfrac{7\pi}{12}\right) = \dfrac{1-\sqrt{3}}{2\sqrt{2}}$

$\cos\left(\dfrac{7\pi}{12}\right) = \dfrac{\sqrt{2}-\sqrt{6}}{4}$

7 If $z = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$ and $w = \cos\left(-\frac{3\pi}{10}\right) + i\sin\left(-\frac{3\pi}{10}\right)$, find $\frac{z^2}{w^5}$ in mod–arg form.

$z = e^{i2\pi/5}$ so $z^2 = e^{i4\pi/5}$

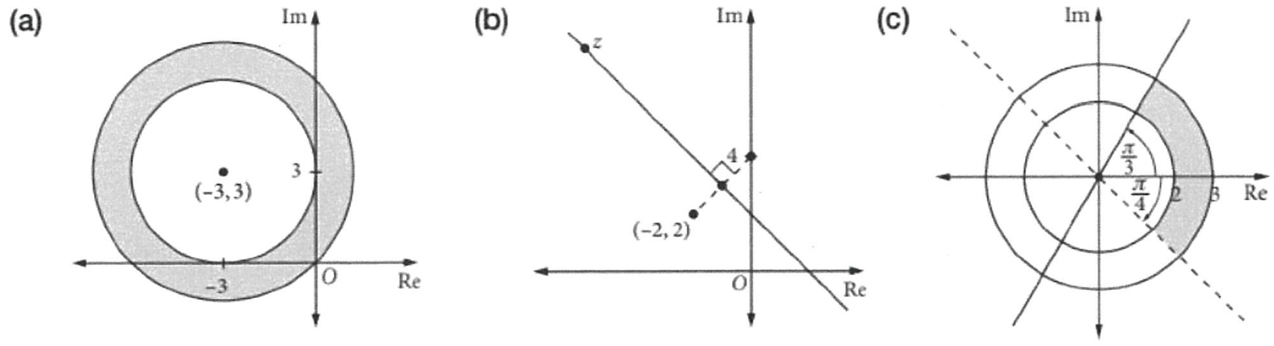
$w = e^{-i3\pi/10}$ so $w^5 = \left(e^{-i3\pi/10}\right)^5 = e^{-i3\pi/2}$

$w^5 = \cos\left(\frac{-3\pi}{2}\right) + i\sin\left(\frac{-3\pi}{2}\right)$

$w^5 = i = e^{i\pi/2}$

so $\frac{z^2}{w^5} = \frac{e^{i4\pi/5}}{e^{i\pi/2}} = e^{i\left(\frac{4\pi}{5} - \frac{\pi}{2}\right)} = e^{i3\pi/10}$

8 Describe each of the following regions of the Argand diagram algebraically.



a) $3 \leqslant |z-(-3+3i)| \leqslant r$ where $r=\sqrt{3^2+3^2}=\sqrt{18}=3\sqrt{2}$

$3 \leqslant |z+3-3i| \leqslant 3\sqrt{2}$

b) $|z-(-2+2i)| = |z-4i|$ (z equidistant of points $(-2+2i)$ and $4i$)

$|z+2-2i| = |z-4i|$

c) $2 \leqslant |z| \leqslant 3$ and $\arg z \in \left[-\frac{\pi}{4}, \frac{\pi}{3}\right]$

(or $-\frac{\pi}{4} \leqslant \arg(z) \leqslant \frac{\pi}{3}$)

10 Find ~~the product of the five 5th roots of~~ $(1+\sqrt{2}i)^3$.

$1+\sqrt{2}i = \sqrt{3}\left(\frac{1}{\sqrt{3}} + \frac{\sqrt{2}}{\sqrt{3}}i\right)$ $\qquad \sqrt{1^2+2} = \sqrt{3}$

No exact values.

$(1+\sqrt{2}i)^3 = 1^3 + 3 \times 1^2 \times \sqrt{2}i + 3 \times 1 \times (\sqrt{2}i)^2 + (\sqrt{2}i)^3$

(using Binomial expansion)
or Pascal's triangle

$(1+\sqrt{2}i)^3 = 1 + 3\sqrt{2}i + 3 \times (-2) + 2\sqrt{2}(-1)i$

$= -5 + i(3\sqrt{2} - 2\sqrt{2})$

$= -5 + \sqrt{2}i$

11 If $z = \cos\theta + i\sin\theta$:

(a) Show that $\arg(z^2 + z^4) = 3\theta$. **(b)** Show that $z^2 + z^4 = 2\cos\theta(\cos 3\theta + i\sin 3\theta)$.

(c) Find the value(s) of θ for which $z^2 + z^4$ is purely imaginary, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

a) $z^2 = e^{i2\theta}$ $\quad z^4 = e^{i4\theta}$

$z^2 + z^4 = e^{i2\theta} + e^{i4\theta} = e^{i3\theta}\left(e^{-i\theta} + e^{i\theta}\right)$

$= e^{i3\theta} \times \underbrace{2\cos\theta}_{\text{real}}$ $\qquad$ But $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

So $\arg(z^2 + z^4) = 3\theta$

b) see a) above.

c) $z^2 + z^4 = 2\cos\theta \; e^{i3\theta}$

is purely imaginary when $\cos 3\theta = 0$

i.e. $3\theta = \pm\frac{\pi}{2} + 2n\pi$

$\theta = \frac{2n\pi}{3} \pm \frac{\pi}{2}$

$n = 0$ $\quad \theta = \pm\pi/2$

$n = 1$ $\quad \theta = \frac{2\pi}{3} \pm \frac{\pi}{2}$ $\qquad \theta = \frac{\pi}{6}$

$n = -1$ $\quad \theta = -\frac{\pi}{6}$

so $\theta = \pm\frac{\pi}{6}$

13 **(a)** If w is a root of $z^{12} = i$, show that $-w$ is also a root.

(b) Let z_1 and z_2 be two distinct roots of $z^{12} = i$. Show that $|z_1 + z_2| < 2$.

a) $w^{12} = i$ $\quad (-w)^{12} = (-1)^{12} \times w^{12} = w^{12}$

So $w^{12} = (-w)^{12} = i$ i.e. $(-w)$ is a root of $z^{12} = i$

b) Using the triangle inequality,

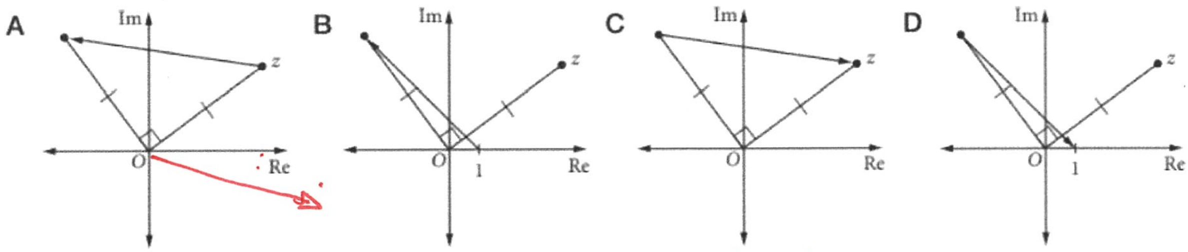
$$|z_1 + z_2| \leqslant |z_1| + |z_2|$$

the roots of $z^{12} = i$ are positioned on the circle centred on O of radius $|i| = 1$ so $|z_1| = |z_2| = 1$

$|z_1 + z_2| \leqslant 1 + 1$ so $|z_1 + z_2| \leqslant 2$

But they ~~are~~ distincts roots so in fact $|z_1 + z_2| < 2$

15 On an Argand diagram, point Z is shown to represent the complex number z. Which diagram below shows the vector that represents $(1 - i)z$?



$$(1-i) = \sqrt{2}\left[\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right] = \sqrt{2}\, e^{i(-\pi/4)}$$

So $(1-i)z = \sqrt{2}\, e^{-i\pi/4} z$

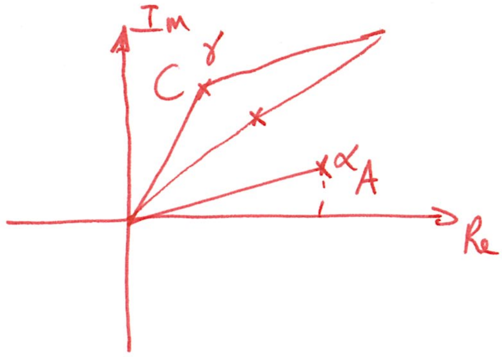
i.e when we multiply the two complex numbers, we increase the modulus of z by a factor $\sqrt{2}$ and we then apply a rotation of $(-\pi/4)$

So $\boxed{C}$

16 On an Argand diagram, the points A, B, C and D represent the complex numbers α, β, γ and δ respectively.

(a) Describe the point that represents $\frac{1}{2}(\alpha+\gamma)$.

(b) If $\alpha+\gamma=\beta+\delta$, deduce that $ABCD$ is a parallelogram.



a) The point represented by $\frac{1}{2}(\alpha+\gamma)$ is the middle of segment A and C

b) if $\alpha+\gamma=\beta+\delta$

then $\frac{1}{2}(\alpha+\beta)=\frac{1}{2}(\beta+\delta)$

So the midpoints of segments AC and BD coincide.

i.e. the diagonals bisect each other.

$\therefore$ ABCD is a parallelogram

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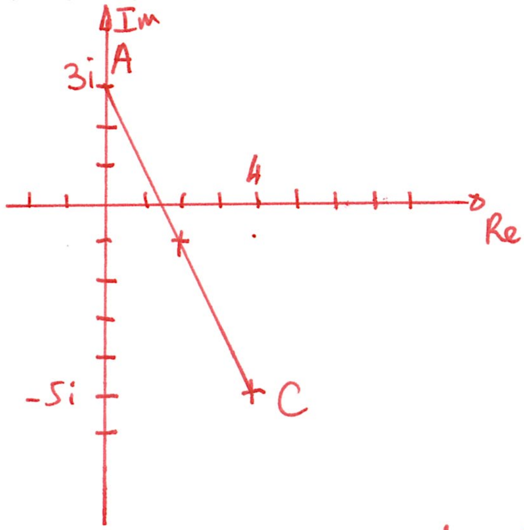
18 On an Argand diagram, the points A and C represent the complex numbers $3i$ and $4-5i$ respectively. $ABCD$ is a rhombus.

(a) Find the Cartesian equation of the diagonal BD.

(b) Show that the diagonal BD is also represented by the equation $(1+2i)z+(1-2i)\bar{z}-8=0$.



a) midpoint of AC is $(2-i)$

gradient of AC is $-\frac{8}{4} = -2$

So gradient of BD is $(\frac{1}{2})$

$$y-(-1) = \frac{1}{2}(x-2)$$

$$y = \frac{1}{2}x - 1 - 1$$ So $y = \frac{1}{2}x - 2$ or $2y - x + 4 = 0$

b) if $z = x + iy$

then $(1+2i)z + (1-2i)\bar{z} - 8 = (1+2i)(x+iy) + (1-2i)(x-iy) - 8$

$$= x + \cancel{iy} + \cancel{2i}x - 2y + x - \cancel{iy} - \cancel{2i}x - 2y - 8$$

$$= 2x - 4y - 8$$

$$= 2\underbrace{(x - 2y - 4)}_{=0}$$ if the point belongs to

the line $x - 2y - 4 = 0$

So if the point belongs to this line, then

$$(1+2i)z + (1-2i)\bar{z} - 8 = 0$$

19 If w is a non-real root of the equation $z^5 = 1$, show that:

(a) $1 + w + w^2 + w^3 + w^4 = 0$ **(b)** $(1 - w)(1 - w^2)(1 - w^3)(1 - w^4) = 5$

(c) $z_1 = w + w^4$ and $z_2 = w^2 + w^3$ are the roots of the quadratic equation $z^2 + z - 1 = 0$.

a) $z^5 - 1 = 0 \iff (z-1)[z^4 + z^3 + z^2 + z + 1] = 0$

So if z is a non real root then $z^4 + z^3 + z^2 + z + 1 = 0$

b) $(1-w)(1-w^2)(1-w^3)(1-w^4) = (1-w)(1-w^4)(1-w^2)(1-w^3)$

$= [1 - w - w^4 + w^5][1 - w^2 - w^3 + w^5]$

but $w^5 = 1$ so

$= [1 - w - w^4 + 1][1 - w^2 - w^3 + 1]$

$= [2 - w - w^4][2 - w^2 - w^3]$

$= 4 - 2w^2 - 2w^3 - 2w + w^3 + w^4 - 2w^4 + w^6 + w^7$

But $w^6 = w^5 \times w = w$ and $w^7 = w^5 \times w^2 = w^2$

$= 4 - 2w^2 - w^3 - 2w - w^4 + w + w^2$

$= 4 \underbrace{- w - w^2 - w^3 - w^4}_{=1} = 5$

c) $z^2 + z - 1 = (w + w^4)^2 + (w + w^4) - 1$

$= w^2 + 2w^5 + \underbrace{w^8}_{= w^3} + w + w^4 - 1$ But $w^5 = 1$, $w^8 = w^3$

$= w + w^2 + w^3 + w^4 + 2 - 1$

$= 1 + w + w^2 + w^3 + w^4 = 0$

likewise for $(w^2 + w^3)$

20 **(a)** Find the cube roots of -8 in mod–arg form.

(b) If w_1 and w_2 are the non-real roots of -8, show that $w_1^{6n} + w_2^{6n} = 2^{6n+1}$ for all integers n.

a) $z^3 = -8 = -2^3 = 2^3 \times (-1) = 2^3 \times e^{-i\pi} = (2e^{-i\pi/3})^3$

So $z = 2e^{-\frac{i\pi}{3} + \frac{i2n\pi}{3}}$

so $z = 2e^{i\pi/3}$ $\quad z = 2e^{i\pi}$ $\quad z = 2e^{-i\pi/3}$

b) $w_1^3 = -8$ and $w_2^3 = -8$

$w_1^{6n} + w_2^{6n} = (w_1^6)^n + (w_2^6)^n = [(w_1^3)^2]^n + [(w_1^3)^2]^n$

$= (-8)^{2n} + 8^{2n}$

$= (-2^3)^{2n} + (2^3)^{2n}$

$= (-1)^{2n}(2^3)^{2n} + (2^3)^{2n}$

$= (2^3)^{2n} + (2^3)^{2n}$

$= 2 \times (2^3)^{2n}$

$= 2^{6n+1}$

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21 On an Argand diagram, A represents the complex number $z = \cos\theta + i\sin\theta$. B represents wz, where $w = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$. M is the midpoint of OB.

(a) Show that $\overrightarrow{AM} = \frac{1}{2}wz - z$.

(b) Show that $\left|\frac{1}{2}wz - z\right| = \sqrt{2-\sqrt{2}}$.

(c) Show that $\arg\left(\frac{1}{2}wz - z\right) = \frac{5\pi}{8} + \theta$.

a) $wz = 2e^{i\pi/4} \times e^{i\theta} = 2e^{i(\theta+\pi/4)}$

The midpoint M of OB is therefore $e^{i(\theta+\pi/4)}$

$\overrightarrow{AM} = \overrightarrow{AO} + \overrightarrow{OM} = -\overrightarrow{OA} + \overrightarrow{OM} = -e^{i\theta} + e^{i(\theta+\pi/4)}$

$\overrightarrow{AM} = -z + \frac{1}{2}wz = e^{i\theta}\left[-1 + e^{i\pi/4}\right]$

b) $\left|\frac{1}{2}wz - z\right| = \left|e^{i\theta}\left(-1+e^{i\pi/4}\right)\right| = \left|e^{i\pi/4} - 1\right|$

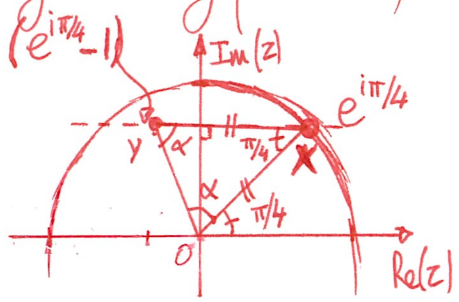
$= \sqrt{\left(\frac{\sqrt{2}}{2} - 1\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2}$

$= \sqrt{\frac{1}{2} - \frac{2}{\sqrt{2}} + 1 + \frac{1}{2}} = \sqrt{2 - \frac{2}{\sqrt{2}}} = \sqrt{2-\sqrt{2}}$

c) So $\frac{1}{2}wz - w = e^{i\theta}\left(e^{i\pi/4} - 1\right)$ and $\left|\frac{1}{2}wz - z\right| = \sqrt{2-\sqrt{2}}$

So $\arg\left(\frac{1}{2}wz - w\right) = \arg\left(e^{i\theta}\left(e^{i\pi/4}-1\right)\right) = \arg e^{i\theta} + \arg\left(e^{i\pi/4} - 1\right)$

$= \theta + \arg\left(e^{i\pi/4} - 1\right)$



But as the diagram shows, OXY is an isosceles triangle, and $\angle YXO$ is $\pi/4$

so $2\alpha + \frac{\pi}{4} = \pi$ (Sum of the 3 interior angles of a triangle is π)

so $\alpha = \frac{1}{2}\left(\frac{3\pi}{4}\right) = \frac{3\pi}{8}$ so $\arg\left(e^{i\pi/4} - 1\right) = \frac{3\pi}{8} + \frac{\pi}{4} = \frac{5\pi}{8}$

$\therefore \arg\left(\frac{1}{2}wz - w\right) = \theta + \frac{5\pi}{8}$

24 **(a)** Given that $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$ (see Example 17, page 16), solve $x^3 - 3\sqrt{3}x^2 - 3x + \sqrt{3} = 0$.

(b) Show that $\tan\frac{\pi}{9} - \tan\frac{2\pi}{9} + \tan\frac{4\pi}{9} = 3\sqrt{3}$.

a) $x^3 - 3\sqrt{3}\,x^2 - 3x + \sqrt{3} = 0$

$\Leftrightarrow \tan^3\theta - 3\sqrt{3}\tan^2\theta - 3\tan\theta + \sqrt{3} = 0$ with $x = \tan\theta$

$\Leftrightarrow \tan^3\theta - 3\tan\theta = 3\sqrt{3}\tan^2\theta - \sqrt{3}$

$\Leftrightarrow 3\tan\theta - \tan^3\theta = \sqrt{3} - 3\sqrt{3}\tan^2\theta = \sqrt{3}\left[1 - 3\tan^2\theta\right]$

$\Leftrightarrow \dfrac{3\tan\theta - \tan^3\theta}{1-\tan^2\theta} = \sqrt{3}$

$\Leftrightarrow \tan 3\theta = \sqrt{3} = \dfrac{\sqrt{3}/2}{1/2}$ so $3\theta = \dfrac{\pi}{3} + n\pi$

$\boxed{\theta = \dfrac{\pi}{9} + n\dfrac{\pi}{3}}$

For $n=0$ $\theta = \pi/9$

For $n=1$ $\theta = \pi/9 + \pi/3 = \dfrac{4\pi}{9}$

$\cancel{\text{for } n=2 \quad \theta = \frac{\pi}{9} + \frac{2\pi}{3} = \frac{7\pi}{9}}$

for $n=-1$ $\theta = \dfrac{\pi}{9} - \dfrac{\pi}{3} = -\dfrac{2\pi}{9}$

So $x_1 = \tan\left(\dfrac{\pi}{9}\right)$

$x_2 = \tan\left(\dfrac{4\pi}{9}\right)$ or $x_3 = -\tan\left(\dfrac{2\pi}{9}\right)$

b) using the sum of roots for a cubic equation:

$\alpha + \beta + \gamma = -\dfrac{b}{a} = \dfrac{-(-3\sqrt{3})}{1} = 3\sqrt{3}$

So $\tan\left(\dfrac{\pi}{9}\right) - \tan\left(\dfrac{2\pi}{9}\right) + \tan\left(\dfrac{4\pi}{9}\right) = 3\sqrt{3}$

30 The polynomial $P(x) = ax^3 + bx + c$ has a multiple zero at $x = 2$ and has a remainder of 20 when divided by $x + 2$. Find a, b and c.

$P(x) = a(x-2)^2(x-k) = a(x^2-4x+4)(x-k)$

$P(x) = a[x^3 - kx^2 - 4x^2 + 4kx + 4x - 4k]$

$P(x) = ax^3 - ax^2(k+4) + 4ax(k+1) - 4ak.$

So $k+4=0 \quad k=-4$

$P(x) = ax^3 - 0 + 4ax(-3) + 16a$

$P(x) = ax^3 - 12ax + 16a$

So $16a = c$ and $-12a = b$

Further $P(-2) = a(-2)^3 - 2b + c = 20$

So $-8a - 2b + c = 20$

$-8a - 2(-12a) + 16a = 20$

$a[-8 + 24 + 16] = 20 \qquad a = \frac{20}{32} = \frac{5}{8}$

and then $b = -12 \times \frac{5}{8} = -\frac{15}{2}$

$c = 16a = 16 \times \frac{5}{8} = 10$

$$\boxed{P(x) = \frac{5}{8}x^3 - \frac{15}{2}x + 10}$$