CURVES AND REGIONS ON THE ARGAND DIAGRAM

We have seen that the modulus and argument, while defined algebraically, can also be thought of geometrically.

If $z = x + iy$, then:

- $|z| = \sqrt{x^2 + y^2}$ is the algebraic definition, while geometrically $|z|$ is the distance from the origin O to the point (x, y)
- algebraically $|z - (a + ib)| = \sqrt{(x - a)^2 + (y - b)^2}$, while geometrically $|z - (a + ib)|$ is the magnitude of the vector from (a, b) to the point representing z
- geometrically, $arg(z)$ is the angle made with the positive direction of the real axis by the vector from the origin O to the point representing z
- geometrically, $arg(z - z_1)$ is the angle made with the positive direction of the real axis by the vector from the point representing z_1 to the point representing z
- $arg(0)$ is undefined

When we need to sketch a subset on an Argand diagram, we need to interpret the information geometrically rather than algebraically, unless the question contains an obvious algebraic substitution. For example, the presence of $z + \bar{z}$ (which equals $2x$) or $z - \bar{z}$ (which equals $2yi$) would suggest an algebraic approach.

Example 33

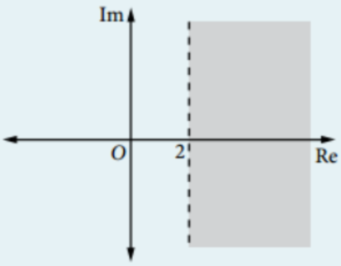
On an Argand diagram, sketch the subsets of z for each of the following.

(a) $z + \bar{z} > 4$ **(b)** $\text{Im}(z) = 3$ **(c)** $2|z| = z + \bar{z} + 4$

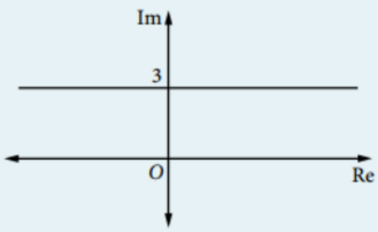
Solution

These need to be done algebraically. Let $z = x + iy$.

(a)
$$z + \bar{z} > 4$$
$$2x > 4$$
$$x > 2$$

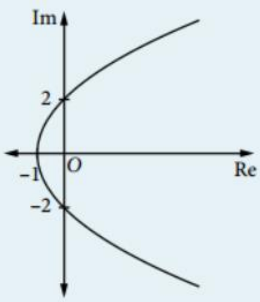


(b)
$$\text{Im}(z) = 3$$
$$y = 3$$



(c)
$$2|z| = z + \bar{z} + 4$$
$$2\sqrt{x^2 + y^2} = 2x + 4$$
$$4x^2 + 4y^2 = 4x^2 + 16x + 16$$
$$y^2 = 4x + 4$$

The region is the set of points on the parabola.



Example 34

On an Argand diagram, sketch the subsets of z for each of the following.

(a) $|z| = 2$ **(b)** $|z + 2 - 2i| = 2$ **(c)** $|z - 2i| = |z + 1 - i|$

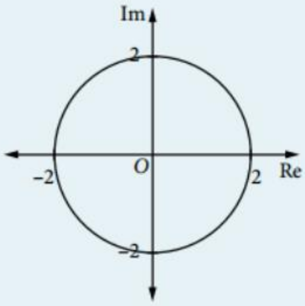
(d) For part **(b)**, find the possible values of $\arg z$ and the maximum and minimum values of $|z|$ on the locus.

Solution

These are best done geometrically, regarding each modulus as a distance. Parts **(a)** and **(c)** are also solved algebraically below (with $z = x + iy$).

(a) Geometrically:

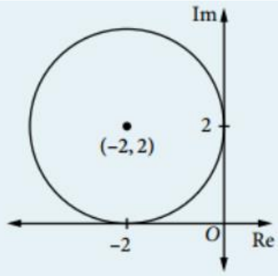
The distance from the origin to the point representing z is equal to 2.
(You can just say 'the distance from O to z is 2'.)
The subset is the circle with centre O and radius 2.



Algebraically:

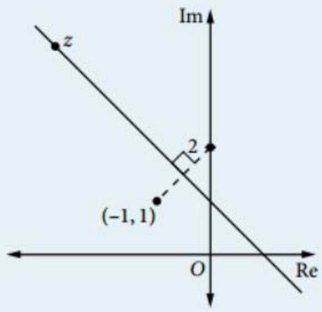
$$\sqrt{x^2 + y^2} = 2$$
$$x^2 + y^2 = 4$$

(b) $|z + 2 - 2i| = 2$
$|z - (-2 + 2i)| = 2$ (written in the form $|z - z_1|$)
The distance from $-2 + 2i$ to z is 2.
The subset is the circle with centre $(-2, 2)$ and radius 2.



(c) Geometrically:

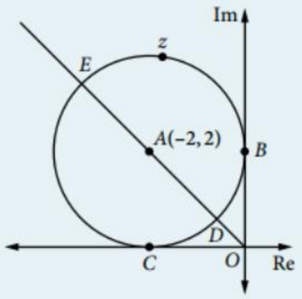
The distance from $2i$ to z is equal to the distance from $-1 + i$ to z,
i.e. z is equidistant from $2i$ and $-1 + i$.
This is the perpendicular bisector of the interval joining $(0, 2)$ and $(-1, 1)$.



Algebraically:

$$|x + (y-2)i| = |(x+1) + (y-1)i|$$
$$\sqrt{x^2 + (y-2)^2} = \sqrt{(x+1)^2 + (y-1)^2}$$
$$x^2 + y^2 - 4y + 4 = x^2 + 2x + 1 + y^2 - 2y + 1$$
$$x + y - 1 = 0$$

(d) As z moves around the circle, the smallest value of $\arg z$ occurs when z is at point B (where $\arg z = \frac{\pi}{2}$). The largest value of $\arg z$ occurs when z is at C (where $\arg z = \pi$).
$\therefore \frac{\pi}{2} \le \arg z \le \pi$



The smallest value of $|z|$ occurs when z is at D and the largest value when z is at E.
Now $OA = 2\sqrt{2}$ (Pythagoras), $AD = 2$ (radius)
$\therefore$ minimum $|z| = OD = OA - AD = 2\sqrt{2} - 2$
maximum $|z| = OE = OA + AE = 2\sqrt{2} + 2$

Example 35

On an Argand diagram, sketch the region of the complex plane for each of the following.

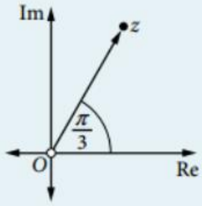
(a) $\arg z = \frac{\pi}{3}$ **(b)** $-\frac{\pi}{4} < \arg z \leq \frac{\pi}{3}$ **(c)** $\arg(z - 2 + 2i) = \frac{3\pi}{4}$

Solution

These can be done geometrically.

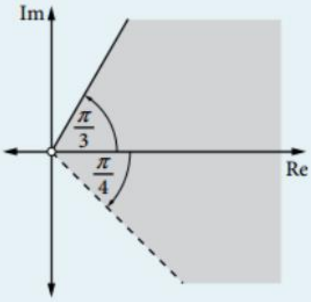
(a) The vector from O to the point representing z is inclined to the positive direction of the real axis at an angle of $\frac{\pi}{3}$.

Note the open circle at O: $z \neq 0$ as $\arg 0$ is undefined.



(b) This is the region between the vectors from O that have inclinations of $-\frac{\pi}{4}$ and $\frac{\pi}{3}$. z can be anywhere in the shaded region.

Note the dashed line on the vector inclined at $-\frac{\pi}{4}$ (due to the $<$ symbol) and the open circle at O (for the undefined $\arg 0$).

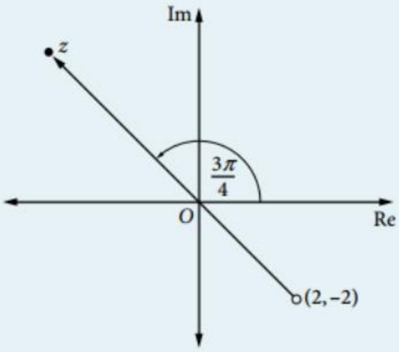


(c) $\arg(z - 2 + 2i) = \frac{3\pi}{4}$

$\therefore \arg(z - (2 - 2i)) = \frac{3\pi}{4}$

The vector from $(2, -2)$ to the point representing z is inclined at $\frac{3\pi}{4}$ to the positive direction of the real axis.

Note that the vector passes through O. (You should try to notice details like this without being prompted.) Also note the open circle at $(2, -2)$, again for the undefined $\arg 0$.



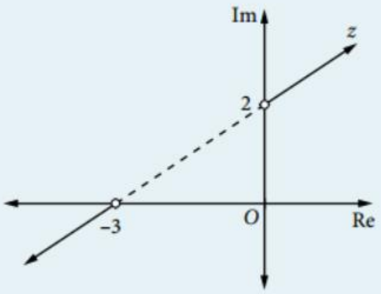
Example 36

On an Argand diagram, sketch the subsets of z for each of the following.

(a) $\arg\left(\frac{z-2i}{z+3}\right)=0$ **(b)** $\arg\left(\frac{z-2i}{z+3}\right)=\pi$

Solution

Use the result $\arg\left(\frac{z_1}{z_2}\right)=\arg z_1-\arg z_2$.



(a) $\arg(z-2i)-\arg(z+3)=0$

$\therefore \arg(z-2i)=\arg(z+3)$

The angle of inclination of the vector from $2i$ to z is equal to the angle of inclination of the vector from -3 to z. This means that z is along the straight line that joins $2i$ and -3, but z must be on the 'outer' parts of the line ('going away'). The direction from $2i$ to z is exactly the same direction as from -3 to z.

Note the open circles at $2i$ and -3.

(b) $\arg(z-2i)-\arg(z+3)=\pi$

The direction from $2i$ to z is exactly opposite to the direction from -3 to z. This means that z is anywhere along the 'inner' part of the straight line that joins $2i$ and -3.

Note the open circles at $2i$ and -3.