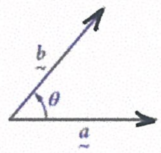
SCALAR PRODUCT OF VECTORS

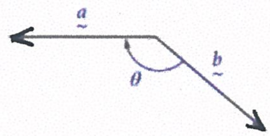
The scalar product (also called the dot product or the inner product) is a way of multiplying two vectors. The result of this multiplication is a scalar quantity (with magnitude but no direction).

The scalar product of two vectors $\vec{a}$ and $\vec{b}$ is written $\vec{a}\,.\,\vec{b}$ (read "$\vec{a}$ dot $\vec{b}$")

There is also another product operation on vectors, the vector product or cross product, which is written $\vec{a} \times \vec{b}$ (read "$\vec{a}$ cross $\vec{b}$"). The result of this multiplication is a vector, not a scalar like for the scalar product. The vector product, which is widely used in Physics, is not studied in this course.

If θ is the angle between the positive directions of two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, then the scalar product is defined to be $\underset{\sim}{a} \bullet \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$.



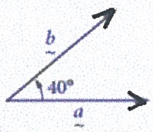


For a straight angle, $\theta = \pi$, so $\underset{\sim}{a} \bullet \underset{\sim}{b} = -|\underset{\sim}{a}||\underset{\sim}{b}|$ as $\cos\pi = -1$.

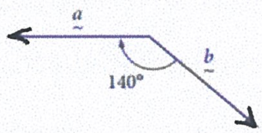
Example 18

Given $|\underset{\sim}{a}| = 5$ and $|\underset{\sim}{b}| = 6$, find the scalar product of $\underset{\sim}{a}$ and $\underset{\sim}{b}$, correct to two decimal places, in each of the following.

(a)



(b)



Solution

(a) $\underset{\sim}{a} \bullet \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$

$= 5 \times 6 \times \cos 40°$

$= 22.98$

(b) $\underset{\sim}{a} \bullet \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$

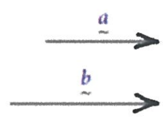
$= 5 \times 6 \times \cos 140°$

$= -22.98$

Special cases of the scalar product

Parallel vectors

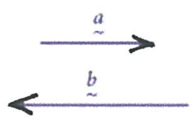
If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel vectors in the same direction, then $\underset{\sim}{a} \bullet \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos 0°$.



As $\cos 0° = 1$, this means: $\underset{\sim}{a} \bullet \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|$

If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel vectors but in opposite directions, then: $\underset{\sim}{a} \bullet \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\pi$

$= -|\underset{\sim}{a}||\underset{\sim}{b}|$



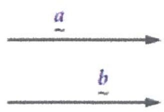
Equal vectors

If $\underset{\sim}{a} = \underset{\sim}{b}$, then $\theta = 0$ (just as for any parallel vectors), so the scalar product is the magnitude squared:

$\underset{\sim}{a} \bullet \underset{\sim}{a} = |\underset{\sim}{a}||\underset{\sim}{a}|\cos 0°$

$= |\underset{\sim}{a}||\underset{\sim}{a}|$

$= |\underset{\sim}{a}|^2$



SCALAR PRODUCT OF VECTORS

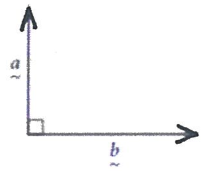
Perpendicular vectors

Two vectors are said to be perpendicular or *orthogonal* if the angle between their directions is a right angle (90°).

If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular vectors, then: $\underset{\sim}{a} \bullet \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\frac{\pi}{2}$

$$= |\underset{\sim}{a}||\underset{\sim}{b}| \times 0$$

$$= 0$$



Also, if $\underset{\sim}{a} \bullet \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta = 0$, then $|\underset{\sim}{a}| = 0$ or $|\underset{\sim}{b}| = 0$ or $\theta = \frac{\pi}{2}$.

This leads to an important property of perpendicular vectors: if the scalar product of two non-zero vectors is zero, then the vectors are perpendicular.

If $\underset{\sim}{a} \bullet \underset{\sim}{b} = 0$ for non-zero vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, then $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular.

An important property of the unit vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$ is that they are perpendicular, so that $\underset{\sim}{i} \bullet \underset{\sim}{j} = \underset{\sim}{j} \bullet \underset{\sim}{i} = 0$.

Also, using the property of parallel vectors: $\underset{\sim}{i} \bullet \underset{\sim}{i} = \underset{\sim}{j} \bullet \underset{\sim}{j} = 1$.

Scalar product for vectors in component form

In component form, for $\underset{\sim}{a} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$ and $\underset{\sim}{b} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$, the scalar product is:

$$\begin{aligned}\underset{\sim}{a} \bullet \underset{\sim}{b} &= \left(x_1\underset{\sim}{i} + y_1\underset{\sim}{j}\right) \bullet \left(x_2\underset{\sim}{i} + y_2\underset{\sim}{j}\right) \\ &= (x_1x_2)(\underset{\sim}{i} \bullet \underset{\sim}{i}) + (x_1y_2)(\underset{\sim}{i} \bullet \underset{\sim}{j}) + (y_1x_2)(\underset{\sim}{j} \bullet \underset{\sim}{i}) + (y_1y_2)(\underset{\sim}{j} \bullet \underset{\sim}{j}) \\ &= x_1x_2 + y_1y_2 \quad \text{as } \underset{\sim}{i} \bullet \underset{\sim}{i} = \underset{\sim}{j} \bullet \underset{\sim}{j} = 1 \text{ and } \underset{\sim}{i} \bullet \underset{\sim}{j} = \underset{\sim}{j} \bullet \underset{\sim}{i} = 0\end{aligned}$$

Consider two vectors, $\underset{\sim}{a} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$ making an angle θ_1 with the x-axis and $\underset{\sim}{b} = x_2\underset{\sim}{i} + y_1\underset{\sim}{j}$ making an angle θ_2 with the x-axis. The angle between the vectors is $\theta = \theta_1 - \theta_2$.

$$\begin{aligned}\underset{\sim}{a} \bullet \underset{\sim}{b} &= |a||b|\cos\theta \\ &= |a||b|\cos(\theta_1 - \theta_2) \\ &= |a||b|(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) \\ &= |a|\cos\theta_1|b|\cos\theta_2 + |a|\sin\theta_1|b|\sin\theta \\ &= x_1x_2 + y_1y_2\end{aligned}$$

For $\underset{\sim}{a} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$ and $\underset{\sim}{b} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$ the scalar product of the vectors in component form is $\underset{\sim}{a} \bullet \underset{\sim}{b} = x_1x_2 + y_1y_2$.

This means there are two basic expressions for the scalar product: $\underset{\sim}{a} \bullet \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta = x_1x_2 + y_1y_2$.

Example 19

Find the scalar product $\underset{\sim}{a} \bullet \underset{\sim}{b}$, given $\underset{\sim}{a} = 2\underset{\sim}{i} + 5\underset{\sim}{j}$ and $\underset{\sim}{b} = -3\underset{\sim}{i} + 4\underset{\sim}{j}$.

Solution

$\underset{\sim}{a} \bullet \underset{\sim}{b} = \left(2\underset{\sim}{i} + 5\underset{\sim}{j}\right) \bullet \left(-3\underset{\sim}{i} + 4\underset{\sim}{j}\right)$

Multiply the coefficients of the like components and sum together: $\underset{\sim}{a} \bullet \underset{\sim}{b} = 2 \times (-3) + 5 \times 4$

$$= 14$$

SCALAR PRODUCT OF VECTORS

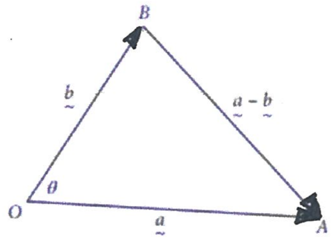
Geometric interpretation of the scalar product

Three vectors form the triangle ΔAOB and the length of each side is the magnitude of the vector forming that side.

The cosine rule states that $|\underset{\sim}{a}-\underset{\sim}{b}|^2=|\underset{\sim}{a}|^2+|\underset{\sim}{b}|^2-2|\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$.

Using the properties of the dot product, the left-hand side can be written as:

$$\begin{aligned}|\underset{\sim}{a}-\underset{\sim}{b}|^2&=(\underset{\sim}{a}-\underset{\sim}{b})\bullet(\underset{\sim}{a}-\underset{\sim}{b})\\&=\underset{\sim}{a}\bullet\underset{\sim}{a}-\underset{\sim}{a}\bullet\underset{\sim}{b}-\underset{\sim}{b}\bullet\underset{\sim}{a}+\underset{\sim}{b}\bullet\underset{\sim}{b}\\&=|\underset{\sim}{a}|^2-2\underset{\sim}{a}\bullet\underset{\sim}{b}+|\underset{\sim}{b}|^2\end{aligned}$$



Rewriting the cosine rule:

$$\begin{aligned}|\underset{\sim}{a}-\underset{\sim}{b}|^2&=|\underset{\sim}{a}|^2+|\underset{\sim}{b}|^2-2|\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta\\|\underset{\sim}{a}|^2-2\underset{\sim}{a}\bullet\underset{\sim}{b}+|\underset{\sim}{b}|^2&=|\underset{\sim}{a}|^2+|\underset{\sim}{b}|^2-2|\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta\\-2\underset{\sim}{a}\bullet\underset{\sim}{b}&=-2|\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta\\\underset{\sim}{a}\bullet\underset{\sim}{b}&=|\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta\end{aligned}$$

Algebraic properties of the scalar product

The scalar product has the following algebraic properties.

1 The commutative law: $\underset{\sim}{a}\bullet\underset{\sim}{b}=\underset{\sim}{b}\bullet\underset{\sim}{a}$

This property follows immediately from the definition $\underset{\sim}{a}\bullet\underset{\sim}{b}=|\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$, since $|\underset{\sim}{a}||\underset{\sim}{b}|=|\underset{\sim}{b}||\underset{\sim}{a}|$.

2 The associative law, including multiplication by a scalar:

$(m\underset{\sim}{a})\bullet\underset{\sim}{b}=m(\underset{\sim}{a}\bullet\underset{\sim}{b})$, where m is a real number.

This property also follows directly from the definition, although it is necessary to distinguish the cases $m<0$ and $m>0$.

3 The distributive law:

$\underset{\sim}{a}\bullet(\underset{\sim}{b}+\underset{\sim}{c})=\underset{\sim}{a}\bullet\underset{\sim}{b}+\underset{\sim}{a}\bullet\underset{\sim}{c}$

The distributive law can be verified using the geometric interpretation of the scalar product and assuming that the projections of $\underset{\sim}{b}$ and $\underset{\sim}{c}$ on $\underset{\sim}{a}$ are both positive.

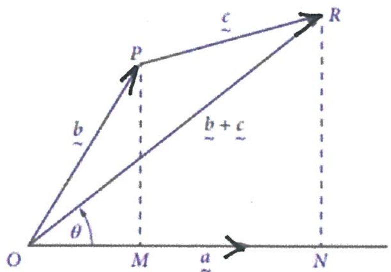
From the diagram: $\underset{\sim}{a}\bullet(\underset{\sim}{b}+\underset{\sim}{c})=|\underset{\sim}{a}||\underset{\sim}{b}+\underset{\sim}{c}|\cos\theta$

$=|\underset{\sim}{a}|\times|\overrightarrow{ON}|$ as $|\underset{\sim}{b}+\underset{\sim}{c}|\cos\theta=|\overrightarrow{ON}|$

Also: $\underset{\sim}{a}\bullet\underset{\sim}{b}+\underset{\sim}{a}\bullet\underset{\sim}{c}=|\underset{\sim}{a}|\times|\overrightarrow{OM}|+|\underset{\sim}{a}|\times|\overrightarrow{MN}|$

$=|\underset{\sim}{a}|\times|\overrightarrow{ON}|$

$=\underset{\sim}{a}\bullet(\underset{\sim}{b}+\underset{\sim}{c})$



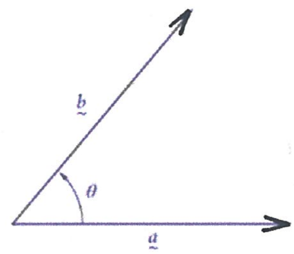
4 An important result involving the distributive law is $(\underset{\sim}{a}+\underset{\sim}{b})(\underset{\sim}{a}-\underset{\sim}{b})=|\underset{\sim}{a}|^2-|\underset{\sim}{b}|^2$:

$$\begin{aligned}(\underset{\sim}{a}+\underset{\sim}{b})\bullet(\underset{\sim}{a}-\underset{\sim}{b})&=\underset{\sim}{a}\bullet\underset{\sim}{a}-\underset{\sim}{a}\bullet\underset{\sim}{b}+\underset{\sim}{b}\bullet\underset{\sim}{a}-\underset{\sim}{b}\bullet\underset{\sim}{b}\\&=\underset{\sim}{a}\bullet\underset{\sim}{a}-\underset{\sim}{b}\bullet\underset{\sim}{b}\text{ since }\underset{\sim}{a}\bullet\underset{\sim}{b}=\underset{\sim}{b}\bullet\underset{\sim}{a}\\&=|\underset{\sim}{a}|^2-|\underset{\sim}{b}|^2\end{aligned}$$

SCALAR PRODUCT OF VECTORS

Finding the angle between two vectors

The scalar product can be used to find the angle between two vectors. Let θ be the angle between the directions of vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ as shown.



$$\underset{\sim}{a} \bullet \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$$

So: $\cos\theta = \dfrac{\underset{\sim}{a} \bullet \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$

$$\therefore \theta = \cos^{-1}\left(\frac{\underset{\sim}{a} \bullet \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}\right)$$

Note: It is usual to consider θ to be the smaller of the two possible angles between the two vectors. Either way, the smaller angle and the larger (reflex) angle will be equivalent for any scalar product, because $\cos(\theta) = \cos(2\pi - \theta)$.

Example 20

Find the angle in degrees, correct to two decimal places, between vectors $\underset{\sim}{a} = 3\underset{\sim}{i} - 2\underset{\sim}{j}$ and $\underset{\sim}{b} = 4\underset{\sim}{i} + \underset{\sim}{j}$.

Solution

Scalar product of the vectors: $\underset{\sim}{a} \bullet \underset{\sim}{b}$: $\underset{\sim}{a} \bullet \underset{\sim}{b} = (3\underset{\sim}{i} - 2\underset{\sim}{j})(4\underset{\sim}{i} + \underset{\sim}{j})$

$$= 3 \times 4 + (-2) \times 1$$

$$= 10$$

Magnitudes of the vectors: $|\underset{\sim}{a}| = \sqrt{3^2 + (-2)^2} = \sqrt{13}$ $\quad |\underset{\sim}{b}| = \sqrt{4^2 + 1^2} = \sqrt{17}$

Substitute into the rule $\theta = \cos^{-1}\left(\dfrac{\underset{\sim}{a} \bullet \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}\right)$ and simplify: $\theta = \cos^{-1}\left(\dfrac{\underset{\sim}{a} \bullet \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}\right)$

$$= \cos^{-1}\left(\frac{10}{\sqrt{13} \times \sqrt{17}}\right)$$

$= 47.73°$ (correct to two decimal places)