THE FIRST DERIVATIVE AND TURNING POINTS

1 A function is given by $f(x) = x^2 - 6x + 8$.

(a) Find $f'(x)$. **(b)** Find the coordinates of any stationary points and determine their nature.
(c) Sketch $y = f(x)$.

a) $f'(x) = 2x - 6$

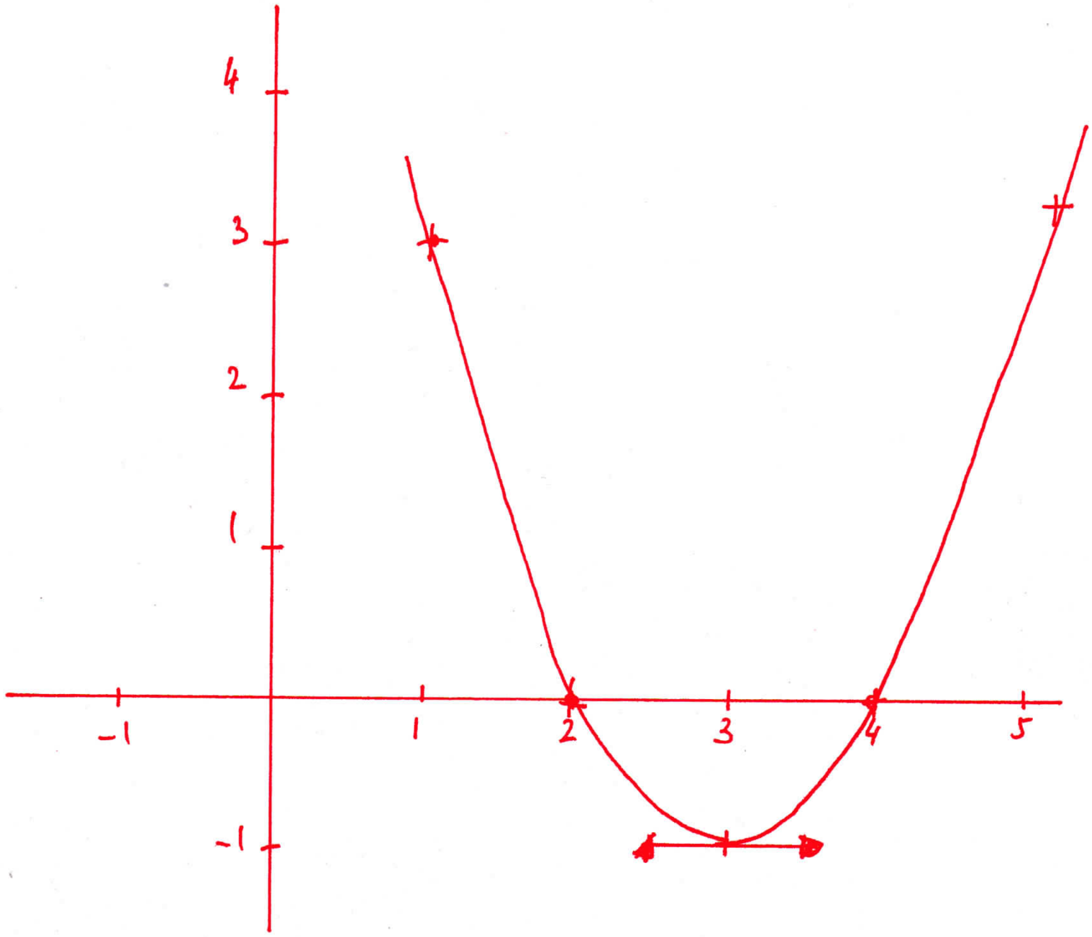
b) Stationary points would fulfil $2x - 6 = 0$

so $x = \frac{6}{2} = 3$ and $f(3) = 3^2 - 6 \times 3 + 8$

$f(3) = -1$

The stationary point is $(3, -1)$

c)



2 If $f'(x) = x^2 - 5x - 6$ then stationary points may occur when:

A $x = 1, -6$ **B** $x = -2, -3$ $\boxed{\textbf{C}}$ $x = -1, 6$ **D** $x = -3, 2$

$x^2 - 5x - 6 = 0$

$\Delta = 25 - 4 \times (-6) \times 1 = 49 = 7^2$

$x_1 = \frac{5-7}{2} = -1$ $x_2 = \frac{5+7}{2} = 6$

5 A function is given by $f(x) = 2x^3 - 15x^2 + 36x$.

(a) Find $f'(x)$. **(b)** Find the coordinates of any stationary points and determine their nature.

(c) Sketch $y = f(x)$.

a) $f'(x) = 6x^2 - 30x + 36$

b) $f'(x) = 0 \Leftrightarrow 6x^2 - 30x + 36 = 0 \Leftrightarrow x^2 - 5x + 6 = 0$

$\Delta = 25 - 4 \times 6 = 1$ $\quad x_1 = \frac{5-1}{2} = 2$ $\quad x_2 = \frac{5+1}{2} = 3$

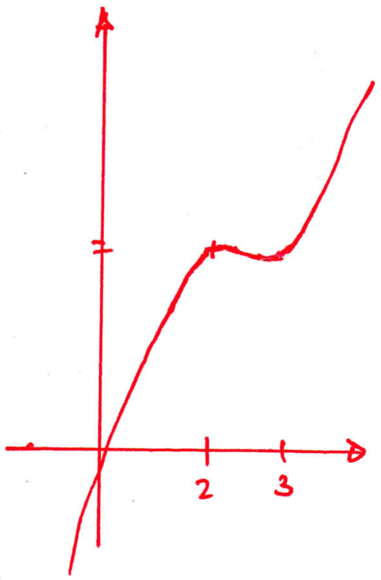
$f(2) = 2 \times 2^3 - 15 \times 2^2 + 36 \times 2 = 28$

$f(3) = 2 \times 3^3 - 15 \times 3^2 + 36 \times 3 = 27$

So stationary points at (2, 28) and (3, 27)

c)

x		2		3	
Sign of $f'(x)$	+	0	−	0	+
Shape	↗		↘		↗



7 Find the maximum value of $5x - 2x^2$. (parabola concave down).

$f'(x) = 5 - 4x$

So $f'(x) = 0$ for $5 - 4x = 0$ $\quad x = 5/4$

$f\left(\frac{5}{4}\right) = 5 \times \frac{5}{4} - 2 \times \left(\frac{5}{4}\right)^2 = \frac{25}{4} - \frac{50}{16} = \frac{25}{8}$

this is the maximum value of $f(x) = 5x - 2x^2$

9 Sketch the curve $y = x^3 - 6x^2$ over the domain $-1 \le x \le 6$, showing the maximum and minimum turning points.

$f'(x) = 3x^2 - 12x = 3x[x-4]$

So f' has two zeros $x = 0$ and $x = 4$

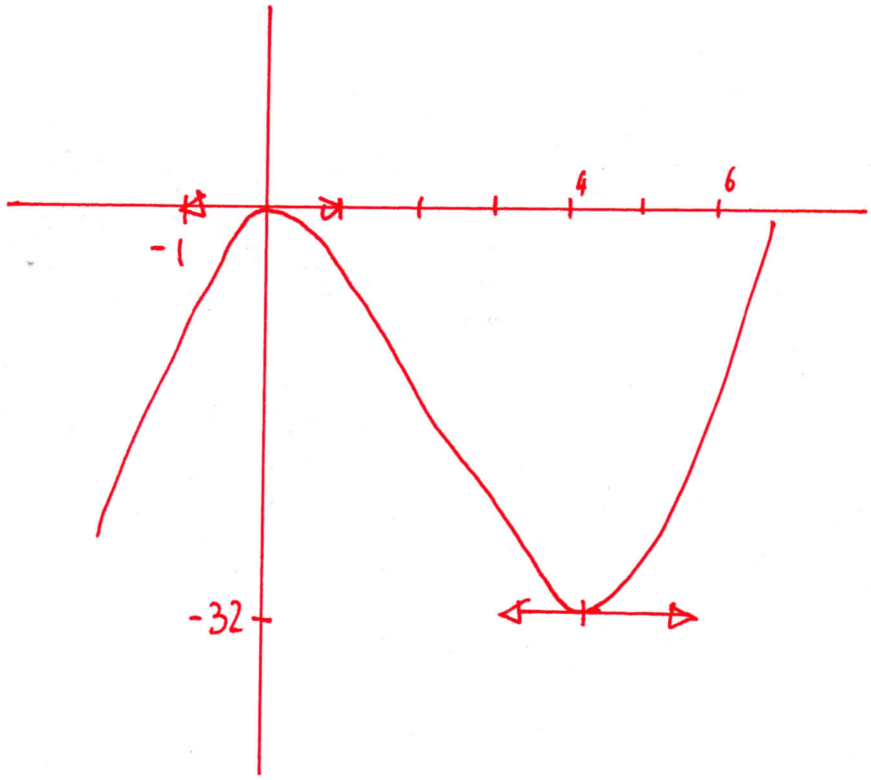
For $x = 0$ $\quad f(0) = 0$

For $x = 4$ $\quad f(4) = 4^3 - 6 \times 4^2 = -32$

x		0		4	
Sign of f'	+	0	−	0	+
Shape of f	$\nearrow$		$\searrow$		$\nearrow$

Local maximum at $(0,0)$

Local minimum at $(4,-32)$.



13 Prove that the parabola $y = ax^2 + bx + c$ has a turning point at $x = \frac{-b}{2a}$.

$f'(x) = 2ax + b$

So there's a turning point when $f'(x) = 0$ or $2ax + b = 0$

$x = -\frac{b}{2a}$

At that point $f\left(\frac{-b}{2a}\right) = a\left(\frac{-b}{2a}\right)^2 + b\left(\frac{-b}{2a}\right) + c$

$f\left(-\frac{b}{2a}\right) = \frac{b^2}{4a} - \frac{b^2}{2a} + c = \frac{b^2 - 2b^2 + 4ac}{4a} = \frac{-b^2 + 4ac}{4a}$

the coordinates of the turning point are $\left(\frac{-b}{2a}, \frac{4ac - b^2}{4a}\right)$

14 Show that the hyperbola $y = \frac{1}{x}$ has no turning points. Also show that its gradient is always negative throughout its domain.

$f(x) = \frac{1}{x} = x^{-1}$ so $f'(x) = (-1)\,x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$

$f'(x) < 0$ so f is decreasing on its domain.

and as $f'(x)$ always different of 0, there is no turning point.